Squaring the Circle and its properties on Hilbert Space

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Abstract

This manuscript contains set properties to define the number $\sqrt{3}r + \delta$; use to prove the equality $A_{Square} = A_{Circle}$ and its use to define the two set of coordinates use on the construction of Squaring the Circle as well as its properties like Hilbert Spaces.

[1]

1 Introduction

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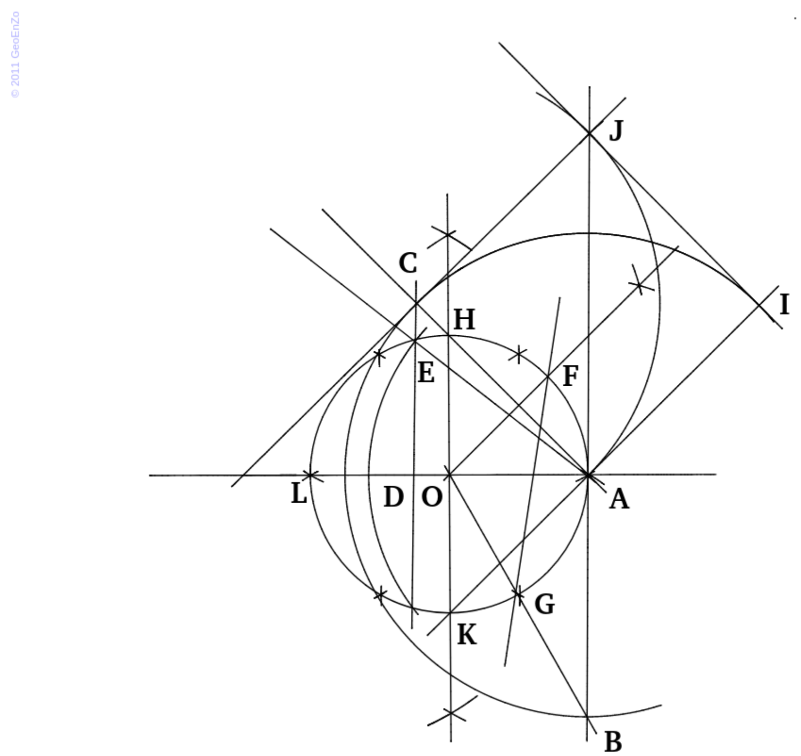

Diagram 1. Squaring the Circle scheme obtained with straightedge and compass

On ref[1] was presented a methodology to obtain through equality; using vector projection; an expression to establish that

Area´s Square = Area´s Circle

Taking only into account the length obtained since measure with compass; the chord $FG = EA$ by translation of length must give

$FG = 2rsin(\frac{\angle FOG}{2})$

with

$\angle FOG = 45 + 60 = 105$

for $r = 1$ we have

$EA = FG = 2(1)sin(\frac{105}{2}) = 1.586706681$

Chord length for segment EL is

$EL = 2rsin(\frac{\angle EOL}{2})$

with

$\angle EOL = 90 - 15 = 75$

hence

$EL = 2(1)sin(\frac{75}{2}) = 1.217522858$

By the other hand the angle $\angle EAD = \angle EAL$ could be obtained using "*Cosine Theorem*"

$c^2 = a^2 + b^2 - 2(a)(b)cos(\angle EAL)$

$EL^2 = EA^2 + LA^2 - 2(EA)(LA)cos(\angle EAL)$

$1.217522858^2 = 1.586706681^2 + 2^2 - 2(1.586706681)(2)cos(\angle EAL)$

$\frac{1.217522858^2 - 1.586706681^2 - 2^2}{-2(1.586706681)(2)} = cos(\angle EAL)$

$arccos(\frac{1.217522858^2 - 1.586706681^2 - 2^2}{-2(1.586706681)(2)}) = \angle EAL$

$\angle EAL = 38.58335963$

Like

$CAcos(\angle CAD) = EAcos(\angle EAL) = EAcos(\angle EAD)$

with $\angle CAD = 45$ hence

$CA = \frac{1.586706681cos(38.58335963)}{cos(45)} = 1.754093209$

Therefore the area of the square link through compass plot to the circle of radius $r = 1$ is

$A_{Square} = 3.0776842984$

and the area of the circle of radius $r = 1$ is

$A_{Circle} = \pi(1)^2 = \pi = 3.1416...$

lets define the ratio α between both areas like

$\alpha = \frac{Area_{Circle}}{Area_{Square}} = \frac{\pi}{3.0776842984} = 1.020765078$

Like the considered angles are constant; hence the length of each linear segment when r increases rise proportionally; therefore α defines a *mechanic constant* that relates the area of both geometric figures (circle and square).

2 The Vector Space Structure of Squaring the Circle

Lets define two sets;

$S :=$ Squaring Set

Where

$S = \{x — AB = \sqrt{3}r + \delta \wedge A_{Square} = A_{Circle}\}$ (see ref[])

and

$S_C :=$ Squaring Compass Set

$S_C = \{x|\ AB = CA = \frac{EAcos(\angle EAD)}{cos(\angle CAD)} \wedge \frac{A_{Circle}}{A_{Square}} = \alpha = cte\}$

Lets give all the points of the circle centered at O and have radius OB´

with $OA = r$ and $CA \in S_C$ by Pythagorean theorem we have;

$r^2 + (\frac{EAcos(\angle EAD)}{cos(\angle CAD)})^2 = (OA)^2$

Hence

$OA = \sqrt{r^2 + (\frac{EAcos(\angle EAD)}{cos(\angle CAD)})^2}$

Remember that any pair of coordinates of a circle centered on the origin have next values

$x = Rcos(\theta)$

$y = Rsin(\theta)$

Hence the coordinate is

$(x, y) = (Rcos(\theta), Rsin(\theta))$

While if the circle is centered at the point (h, k) hence

$x = Rcos(\theta) - h$

$y = Rsin(\theta) - k$

Hence the coordinate is

$(x, y) = (Rcos(\theta) - h, Rsin(\theta) - k)$

if $R = OA$ all the points on the circle required are described.

Now lets give all the points that represent the circle centered at O and have radius OB

With $OA = r$ and $AB = \sqrt{3}r + \delta$

we have for the hypotenuse

$OB = \sqrt{r^2 + (\sqrt{3}r + \delta)^2}$

Like

$\delta = r(\sqrt{\pi} - \sqrt{3})$

therefore

$OB = \sqrt{r^2 + \pi r^2}$

simplifying we have

$OB = (\sqrt{\pi + 1})r$

Lets use the property 1 found on the section "*Some Properties About Abstract Geometrical Objects*"

hence by

$r < d \leq \pi$

adding r for each term we have

$r < d < d + r \leq \pi + r$

using $r = 1$ and applying the methodology used on property 2 we have

$\sqrt{\pi + 1} < \pi + 1$

on property 2 we do not define an upper of lower bound; instead we required that

$r < \alpha < d$

where α is the square root used to implement such methodology about property 2

like $\sqrt{\pi + 1} = 2.0350...$

hence the least natural number to define last inequality is $r = 2$

therefore with $r = 2$ we are going to have

$r < \sqrt{\pi + 1} < d$

later any number between r and d could be obtained with

$r < (\sqrt{\pi + 1})t < d$

with with k

$\frac{r}{\sqrt{\pi+1}} < t < \frac{d}{\sqrt{\pi+}}$

When t represent the radius r; $(t = r)$ of a circle hence property 1 and 2 gives place to the set of circles where "squaring the circle" is possible and have like equations those shown on ref[1].

On this case the coordinates of every point of the circle are

$(x, y) = (\sqrt{r^2 + (\sqrt{3}r + \delta)^2}cos(\theta) - h, \sqrt{r^2 + (\sqrt{3}r + \delta)^2}sin(\theta) - k)$

When $\theta = arctan(\frac{1}{\sqrt{\pi}})$

With $R = \sqrt{r^2 + (\sqrt{3}r + \delta)^2}$

Can be verified manually (with calculator that)

$Rcos(\theta) = \sqrt{3}r + \delta$

$Rsin(\theta) = r$

Now lest prove that S is a Hilbert space

Property A. S and S_C are Hilbert Spaces

Def.

Lets name $\vec{v} = (x, y) \in \Re^2$

$(x, y) = (Rcos(\theta) - h, Rsin(\theta) - k)$

and R, h and k defined as before; note that to be real numbers; i.e. $x, y \in \Re$ hence next properties about a Hilbert space are satisfied using like inner product definition of dot product; because commutative property maintains.

* *Conjugate symmetry*

$< \vec{v}_2, \vec{v}_1 >= \overline{< \vec{v}_1, \vec{v}_2 >}$

* *Linear*

$a\vec{v}_1 + b\vec{v}_2, \vec{v}_3 = a\vec{v}_1, \vec{v}_3 + b\vec{v}_2, \vec{v}_3$

* *Definite Positive*

like $x \vee y > 0$ always for $\vec{v} = (x, y)$

$\vec{v}, \vec{v} > 0$

* *Norm*

$||\vec{v}|| = \sqrt{\vec{v}, \vec{v}}$

Like before; the norm to be define with the inner euclidean inner product the next properties are satisfied.

* *Non negative*

$||\vec{x}|| \geq 0$; $\forall \vec{v} \in V$; else $||\vec{x}|| = 0$ iff $\vec{x} = 0$

* *Homogeneity*

$||k\vec{v}|| = |k| \cdot ||\vec{v}||; \forall \vec{v} \in V; \forall k \in K$

* *Inequality*

$||\vec{v}_1 + \vec{v}_2|| \leq ||\vec{v}_1|| + ||\vec{v}_2||; \forall v_1, v_2 \in V$

Def. A normed vector space is a vector space equipped with a norm, which is a function that measures the length of vectors. Any normed vector space can be equipped with a metric in which the distance between two vectors $\vec{v}_1$ and $\vec{v}_2$ is given by

$d(\vec{v}_1, \vec{v}_2) = ||\vec{v}_1 - \vec{v}_2||$

The metric d is said to be induced b the norm $|| \cdot ||$

Like all the coordinates of the considered vectors are on the circle; the distance between them must be considered through the subtraction of vectors.

By the other hand to define relation between dot product and arc length on the circle; lets use next identities:

$\vec{v}_1 \cdot \vec{v}_2 = \Sigma_{i=1}^{n} v_{1i} v_{2i}$

and

$\vec{v}_1 \cdot \vec{v}_2 = ||\vec{v}_1|| ||\vec{v}_2|| cos\theta$
matching we have

$\Sigma_{i=1}^{n} v_{1i} v_{2i} = (\sqrt{\Sigma_{i=1}^{n} v_{1i}^2})(\sqrt{\Sigma_{i=1}^{n} v_{2i}^2}) cos\theta$

clearing $cos\theta$

$\frac{\Sigma_{i=1}^{n} v_{1i} v_{2i}}{\sqrt{\Sigma_{i=1}^{n} v_{1i}^2}\sqrt{\Sigma_{i=1}^{n} v_{2i}^2}} = cos\theta$
$arccos\{\frac{\Sigma_{i=1}^{n} v_{1i} v_{2i}}{\sqrt{\Sigma_{i=1}^{n} v_{1i}^2}\sqrt{\Sigma_{i=1}^{n} v_{2i}^2}}\} = \theta$

Arc length on a circle can be calculated with

$d_s = R\theta$

$\theta = \frac{d_S}{R}$
substituting

$\frac{ds}{R} = arccos\{\frac{\Sigma_{i=1}^{n} v_{1i} v_{2i}}{\sqrt{\Sigma_{i=1}^{n} v_{1i}^2}\sqrt{\Sigma_{i=1}^{n} v_{2i}^2}}\}$

$ds = R\ arccos\{\frac{\Sigma_{i=1}^{n} v_{1i} v_{2i}}{\sqrt{\Sigma_{i=1}^{n} v_{1i}^{2}}\sqrt{\Sigma_{i=1}^{n} v_{2i}^{2}}}\}$

We have defined the properties of inner product and norm in terms of dot product and polar coordinates i.e. when

$\vec{v} = (x, y) \in V$

$(x, y) = (Rcos(\theta) - h, Rsin(\theta) - k)$

In specific when R is equal to:

$OA \in S \wedge OA \in S_C$

Taking into consideration all properties early mentioned; S and S_C are Hilbert spaces.

Construction of Hilbert space with S and S_C by direct sum $\bigoplus$

Def. Two Hilbert spaces H_1 and H_2 can be combined into another Hilbert space, called the *(orthogonal) direct sum*, and denoted

$H_1 \oplus H_2$

consisting of the set of all ordered pairs (v_{1x}, v_{2x}) and (v_{1y}, v_{2y}) where $v_{ix}, v_{iy} \in H_i$, $i = 1, 2$, and inner product defined by

$(v_{1x}, v_{2x}), (v_{1y}, v_{2y})_{H_1 \oplus H_2} = v_{1x}, v_{1y}{}_{H_1} + v_{2x}, v_{2y}{}_{H_2}$

Proof.

Lets take

$v_{1x}, v_{1y} \in S$ and $v_{2x}, v_{2y} \in S_C$

and lets add both inner products;

$v_{1x}, v_{1y}{}_S + v_{2x}, v_{2y}{}_{S_C} \in S \cup S_C$

Like S and S_C are Hilbert spaces and using last definition about direct product we have that addition is an inner product for $S \cup S_C$ hence

$(v_{1x}, v_{2x}), (v_{1y}, v_{2y})_{S \cup S_C} = v_{1x}, v_{1y}{}_S + v_{2x}, v_{2y}{}_{S_C}$

naming $S \cup S_C$ like $S_1 \oplus S_2$ respectively we have

$(v_{1x}, v_{2x}), (v_{1y}, v_{2y})_{S_1 \oplus S_2} \in \bigoplus S_i$ with $i \in \{1, 2\}$

Under this conception the set S_C contain the coordinates of the scheme of ref[1] made with compass and straightedge; while the set S contain the coordinates of the scheme made with computer where $A_{Square} = A_{Circle}$ for same reference.

3 Some Properties About Abstract Geometrical Objects

Lets consider next diagram:

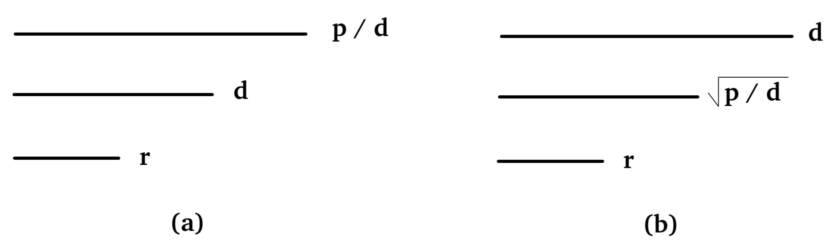

Diagram 2. Properties of a Circle; where the length of linear segments is named with p for the perimeter; d for the diameter and r for the radius of a circle

Lets establish the **upper bounded set** define with $\frac{p}{d} = \pi$

Property 1. π is a supremum for the set of circle or any radius.

A circle can be described through its properties; essentially is formed for just one plot, describing the perimeter (p); however every circle must necessarily to be define through the value of its radius (r).

Lets note that each plot; perimeter and radius could be linear segments with defined length; diagram (a) describes such situation and can be written like next inequality:

$r < p$

An special property can be established for every circle if we take into account the relation through quotient between perimeter and twice the length of its radius (i.e. $2r$); also called diameter of the circle.

Such property gives place to a constant named "pi" and its represented through next mathematical equation:

$\frac{p}{d} = \frac{2\pi r}{2r} = \pi$

Where the formula for perimeter has been used ($p = 2\pi r$).

Next relations also maintains for every circle with a diameter with value: $d \leq \pi$

$r < 2r = d$

$d < \pi$

Therefore

$r < d < \pi$

Such type of circle form a set defined through the value of its radius; where π is an upper bound like will be proof next:

Definition. An *upper bound* of a subset S of a partially ordered set $(P, \leq)$ is an element b of P such that

$b \leq x$ for all $x \in S$

Lets consider the inequality

$0 < r < d < \pi$

Like $d = 2r$ hence

$0 < r < 2r < \pi$

If

$r \leq \frac{\pi}{2}$

Hence

$d \leq \pi$

Therefore

$r < d \leq \pi$

i.e. π an upper bound for the set of circles with radius $r \leq \frac{\pi}{2}$

Like a circle is define through its radius $\forall r \in \Re$

Hence $\pi \leq z$ for all upper bounds z that could be given to define another kind of circle´ set; i.e. π is a *supremum.*

Property 2. $\sqrt{\pi} \in C_\pi$; the circle set of pi when its supremum and fulfills $r < \sqrt{\pi} < d$.

lets take

consider next property about real numbers

$\sqrt{b} < b$

i.e. the square root of every number is less than itself.

If $b = \pi$ hence

$\sqrt{\pi} < \pi$

On the set of circles where π is the supremum we have

$r < d \leq \pi$

Like

$d = \pi$

hence

$\sqrt{\pi} < d$

like

$r \leq \frac{\pi}{2} = 1.5707...$

and

$\sqrt{\pi} = 1.7724...$

$r < \sqrt{p}$

Therefore

$r < \sqrt{\pi} < d$

Like is represented on diagram (b)

Property 3. Iterative addition of r and radicalization of each result of properties 1 and 2 establish a pattern between line segments that defines perimeter, diameter and radius.

Definition. A lower bound of a subset S of a partially ordered set $(P, \leq)$ is an element a of P such that

$a \leq x$ for all $x \in S$

-o-

From property 2 we know that

$r < \sqrt{\pi} < d$ (*)

if

$r = \sqrt{\pi}$

hence

$r \leq \sqrt{\pi}$

therefore

$\sqrt{\pi}$ is an *infimum*

since equality (*) adding r from each term we have

$r + r < \sqrt{\pi} + r < d + r$

i.e.

$r < d < \sqrt{\pi} + r < d + r$

using the infimum $r = \sqrt{\pi}$

we have for the diameter

$d \leq 2\sqrt{\pi}$

i.e.

$r < d \leq \sqrt{\pi} + r$

with

$\sqrt{\pi} + r$ an upper bound for the circle with radius $r \leq \sqrt{\pi}$

Now like we´ve did before on property 2 lets define the square root of the second supremum obtained.

for every real number we have $\sqrt{b} < b$ hence for the second supremum

$\sqrt{\sqrt{\pi} + r} < \sqrt{\pi} + r$

i.e.

1.8827... < 3.5449...

with $r = \sqrt{\pi} = 1.7724...$

and $d = 2r = 2\sqrt{\pi} = 3.5449...$

Hence

$r < \sqrt{\sqrt{\pi} + r} < d$

for the second set defined with the second supremum.

Like before $\sqrt{\sqrt{\pi} + r}$ could be a second infimum if becomes equal to r.

Now adding r to each term we could have from last inequality

$r < d < r + \sqrt{\sqrt{\pi} + r}$

Following same procedure since the proposed for property 1 and 2 we could continue same methodology established for this property and obtain the pattern shows on diagram 3 and next diagram.

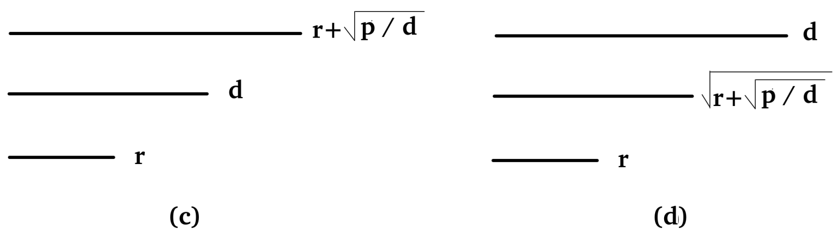

Diagram 3. Pattern of length for each linear segment obtained on property 3.

The sets defined on properties 1, 2 and 3 using supremum and infimum establish a description of a geometrical object; being in some way like an "abstract geometrical object of study".

Property 4. The limit of the iteration method for $r = 1$ give like result ϕ := the golden number.

Lets use r = 1 on the procedure given before. Next table was made on LibreOffice Calc.

n Square Root Addition r =1

1 1.77245385090552 2.77245385090552
2 1.66506872257739 2.66506872257739
3 1.63250382008049 2.63250382008049
4 1.62249925118026 2.62249925118026
5 1.61941324286924 2.61941324286924
6 1.61846014559186 2.61846014559186
7 1.61816567309774 2.61816567309774
8 1.61807468093959 2.61807468093959
9 1.61804656327919 2.61804656327919
10 1.61803787448848 2.61803787448848
11 1.61803518950871 2.61803518950871
12 1.61803435980473 2.61803435980473
13 1.61803410341214 2.61803410341214
14 1.61803402418248 2.61803402418248
15 1.61803399969916 2.61803399969916
16 1.61803399213341 2.61803399213341
17 1.61803398979546 2.61803398979546
18 1.61803398907299 2.61803398907299
19 1.61803398884974 2.61803398884974
20 1.61803398878075 2.61803398878075

21 1.61803398875943 2.61803398875943
22 1.61803398875284 2.61803398875284
23 1.61803398875081 2.61803398875081
24 1.61803398875018 2.61803398875018
25 1.61803398874998 2.61803398874998
26 1.61803398874992 2.61803398874992
27 1.6180339887499 2.6180339887499
28 1.6180339887499 2.6180339887499
29 1.6180339887499 2.6180339887499
30 1.6180339887499 2.61803398874989
31 1.6180339887499 2.61803398874989
32 1.6180339887499 2.61803398874989

the golden number is

$\phi = \frac{1+\sqrt{5}}{2}$

$= 1.6180339887499$

i.e. since term 27 of iteration of square root procedure given before can be obtained.

Take into account that the term
m
$\sqrt{\sqrt{\pi}+1}$ is equivalent to the term 2 of iteration using square root.

the inequality that is related with such term when r = 1

$r < \sqrt{\sqrt{\pi}+1} < d$

and any number between r and d could be define like

$r < (\sqrt{\sqrt{\pi}+1})t < d$

4 References

[1] Diaz, J. R. A. M. (2024). Study by Vector Projection of a Construction Made with a Rule and Compass to Understand Squaring the Circle.